The Fishing Project

Maintained by Jordan Watts

October 28, 2018

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1. Differentiable Manifolds

1.1. Charts, atlases, and manifolds

1.1.1 Manifolds are topological spaces which are locally modeled on euclidean space by homeomorphisms called charts. If such a manifold is covered by charts in a way that the transition maps between these charts all lie in a particular differentiability class one obtains a differentiable manifold. Let us make this more precise.

1.1.2 Definition Assume that $\Bbbk$ is the field of real or complex numbers, and M a topological space. By a *chart* of M one understands a homeomorphism $x : U \to \widetilde{U}$ from an open subset $U \subset M$ onto an open subset $\widetilde{U}$ of some $\Bbbk^n$. We often denote a chart of M by $(x, U, \Bbbk^n)$ or just shortly by (x, U), and call the chart *real* respectively *complex* depending on whether $\Bbbk = \mathbb{R}$ or $\Bbbk = \mathbb{C}$. The number n is called the *real* respectively the *complex dimension* of the chart.

Let $k \in \mathbb{N} \cup \{\infty, \omega\}$. Two real charts $x : U \to \widetilde{U} \subset \mathbb{R}^n$ and $y : V \to \widetilde{V} \subset \mathbb{R}^m$ are said to be $\mathcal{C}^k$-*compatible* if the transition maps $x \circ y^{-1} : y(U \cap V) \to x(U \cap V)$ and $y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$ are both of class $\mathcal{C}^k$, which means continuous if $k = 0$, k-times continuously differentiable if $k \in \mathbb{N}^*$, smooth if $k = \infty$, and real analytic if $k = \omega$. A set $\mathfrak{A}$ of pairwise $\mathcal{C}^k$-compatible charts of M is called a $\mathcal{C}^k$-*atlas of* M or an *atlas of class* $\mathcal{C}^k$ if the charts in $\mathfrak{A}$ cover M which means that for each point $p \in M$ there is a chart $(x, U) \in \mathfrak{A}$ with $p \in U$.

Two complex charts $x : U \to \widetilde{U} \subset \mathbb{C}^n$ and $y : V \to \widetilde{V} \subset \mathbb{C}^m$ are called *holomorphically compatible* if the transition maps $x \circ y^{-1} : y(U \cap V) \to x(U \cap V)$ and $y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$ are both holomorphic. A set $\mathfrak{A}$ of pairwise holomorphically compatible charts of M is a *holomorphic atlas* of M or an *atlas of class* $\mathcal{O}$ if the charts in $\mathfrak{A}$ cover M.

1.1.3 The sets of $\mathcal{C}^k$-atlases and of holomorphic atlases of a topological space M are ordered by set-theoretic inclusion. Obviously, if $\mathfrak{A}$ is an atlas of class $\mathcal{C}^k$ or $\mathcal{O}$ there exists a unique maximal atlas $\widehat{\mathfrak{A}}$ of the same class containing $\mathfrak{A}$. The atlas $\widehat{\mathfrak{A}}$ is obtained from $\mathfrak{A}$ by adding all charts which are $\mathcal{C}^k$-compatible respectively holomorphically compatible with each chart contained in $\mathfrak{A}$.

If $f : N \to M$ is a homeomorphism between two topological spaces N and M, and $\mathfrak{A}$ a maximal atlas on M, the set of charts

$$f^*\mathfrak{A} := \left\{ \left(x \circ f_{|f^{-1}(U)}, f^{-1}(U), \Bbbk^n \right) \mid (x, U, \Bbbk^n) \in \mathfrak{A} \right\}$$

is a maximal atlas on N and of the same differentiability class as $\mathfrak{A}$. In case N comes equipped with a maximal atlas $\mathfrak{B}$, too, the homeomorphism f is called a *diffeomeorphism* from $(N, \mathfrak{B})$ to $(M, \mathfrak{A})$ if $f^*\mathfrak{A}$ coincides with $\mathfrak{B}$. If $\mathfrak{A}$ and $\mathfrak{B}$ are two maximal atlases on M and there exists a homeomorphism $f : M \to M$ which is a diffeomorphism from M equipped with $\mathfrak{A}$ to M equipped with $\mathfrak{B}$, one calls the two atlases $\mathfrak{A}$ and $\mathfrak{B}$ *equivalent*. By definition it is clear that equivalence of maximal atlases on a topological space M is an equivalence relation indeed.

1.1.4 Definition A topological space M is called *locally euclidean*, if for every $p \in M$ there exists an open neighborhood U together with a homeomorphism $x : U \to \widetilde{U}$ mapping U onto an open subset $\widetilde{U} \subset \mathbb{R}^n$ of some euclidean space.

1.1.5 Definition Assume M to be a topological space, and $k \in \mathbb{N} \cup \{\infty, \omega\}$. By a $\mathcal{C}^k$-*structure* or *differentiable structure of class* $\mathcal{C}^k$ on M one understands a maximal $\mathcal{C}^k$-atlas on M. A *holomorphic structure* on M is given by a maximal holomorphic atlas. One sometimes calls a $\mathcal{C}^0$-structure a *locally euclidean structure*, a $\mathcal{C}^\infty$-structure a *smooth structure*, and a $\mathcal{C}^\omega$-structure a (*real*) *analytic structure* on the underlying topological space M.

1.1.6 Remark 1. A topological space M possesses a locally euclidean structure if and only if the space is locally euclidean.

2. John Milnor showed in his paper **??** titled *On manifolds homeomorphic to the* 7*-sphere* that the 7-dimensional sphere admits a smooth structure which is not equivalent to the standard smooth structure defined in **??**. In other words this means that

   Following Milnor, a topological space equipped with a smooth structure which is homeomorphic but not diffeomeorphic to a standard sphere is called an *exotic sphere*. In joint work with Michel Kervaire, Milnor classified in **??** all smooth structures on the 7-sphere up to equivalence and proved that there are exactly 28 of them.

2. Lie theory

2.1. Lie groups

2.1.1 Definition A real or complex manifold G carrying a group structure is called a (*real* respectively *complex*) *Lie group* if the manifold and group structures are compatible in the following sense:

(LieGrp1) The product map $m : G \times G \to G$, $(g, h) \mapsto g \cdot h$ is smooth respectively holomorphic in the complex case.

(LieGrp2) The inversion map $i : G \to G$, $g \mapsto g^{-1}$ is smooth respectively holomorphic in the complex case.

2.1.2 Example The field of real numbers $\mathbb{R}$ together with its standard smooth structure and addition as group operation is a real Lie group. Likewise, the field of complex numbers with its standard complex structure and addition as group operation is a complex Lie group. The pointed spaces $\mathbb{R}^*$ and C^* together with multiplication of real or complex numbers as product are Lie groups as well, the first one being real, the second complex.

3. Symplectic Geometry

3.1. Symplectic linear algebra

4. Geometry of Poisson Manifolds

4.1. Poisson structures

4.1.1 Definition A *Poisson structure* or *Poisson bracket* on a manifold P is an $\mathbb{R}$-bilinear map

$$\{\cdot,\cdot\} : \mathcal{C}^\infty(P) \times \mathcal{C}^\infty(P) \to \mathcal{C}^\infty(P)$$

having the following three properties:

(PS1) The bracket $\{\cdot,\cdot\}$ is *antisymmetric*, i.e.

$$\{f,g\} = -\{g,f\} \quad \text{for all } f,g \in \mathcal{C}^\infty(P).$$

(PS2) *Jacobi's identity* holds true, i.e.

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0 \quad \text{for all } f,g,h \in \mathcal{C}^\infty(P).$$

(PS3) The bracket $\{\cdot,\cdot\}$ fulfills *Leibniz' rule* in its first argument, i.e.

$$\{fg,h\} = f\{g,h\} + \{f,h\}g \quad \text{for all } f,g,h \in \mathcal{C}^\infty(P).$$

If only Axioms **??** and **??** are satisfied, one calls $\{\cdot,\cdot\}$ an *almost Poisson structure* on M. A manifold P together with a Poisson structure (resp. almost Poisson structure) $\{\cdot,\cdot\}$ on it is called a *Poisson manifold* (resp. an *almost Poisson manifold*).

By antisymmetry, a Poisson bracket fulfills Leibniz' rule in its second argument, too. Axioms **??** and **??** imply that $\mathcal{C}^\infty(P)$ equipped with a Poisson bracket $\{\cdot,\cdot\}$ is a Lie algebra. Note that property **??** entails that for fixed $h \in \mathcal{C}^\infty(P)$ the map $\{\cdot,h\}$ is a derivation of the ring $\mathcal{C}^\infty(P)$, hence there exists a smooth vector field $X_h \in \mathfrak{X}(P)$ such that for any $f \in \mathcal{C}^\infty(P)$ the relation $X_h f = \{f,h\}$ holds true. This vector field is called the *hamiltonian vector field* of h. If one of the functions in the bracket is constant, then the Leibniz rule demands that the bracket be equal to zero.

5. Groupoids

5.1. The category of groupoids

5.1.1 A groupoid can be thought of as a generalized group in which only certain multiplications are possible. It can be consisely defined as a small category with all arrows invertible, but the following - logically equivalent - definition reflects the underlying concepts more clearly.

5.1.2 Definition A *groupoid* over a set X is a set G together with the following structure maps:

- a pair of maps $\mathsf{G} \underset{s}{\overset{t}{\rightrightarrows}} X$, where t is called the *target map* while s is called the *source map*,
- a (*partial*) *multiplication* or *product* $m : \mathsf{G}^{(2)} \to \mathsf{G}$, $(g,h) \mapsto gh$ defined on the set of *composable pairs*
$$\mathsf{G}^{(2)} := \{(g,h) \in \mathsf{G} \times \mathsf{G} \mid s(g) = t(h)\} ,$$
- an embedding $u : X \to \mathsf{G}$, called the *identity section* or *unit*, and
- an *inversion map* $i : \mathsf{G} \to \mathsf{G}$, $g \mapsto g^{-1}$.

These data must have the following properties:

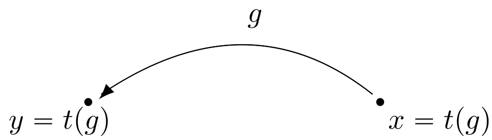
(Grpd1) Multiplication from the right preserves the target, multiplication from the left the source, i.e. $t(gh) = t(g)$ & $s(gh) = s(h)$ for all $(g,h) \in \mathsf{G}^{(2)}$.

(Grpd2) Multiplication is associative, i.e. $(gh)k = g(hk)$ for all $(g,h,k) \in \mathsf{G}^{(3)}$, where
$$\mathsf{G}^{(3)} := \{(g,h,k) \in \mathsf{G} \times \mathsf{G} \times \mathsf{G} \mid s(g) = t(h), s(h) = t(k)\}$$
denotes the set of *composable triples*.

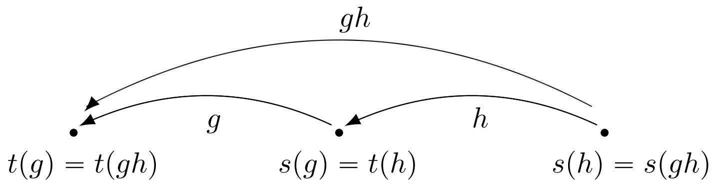
(Grpd3) The unit map acts as identity, i.e. $u(t(g))g = g = gu(s(g))$ for all $g \in \mathsf{G}$. In particular, $t \circ u = s \circ u$ is the identity map on X.

(Grpd4) The inversion map acts by inversion, i.e. $i(g)g = u(s(g))$ & $gi(g) = u(t(g))$ for all $g \in \mathsf{G}$.

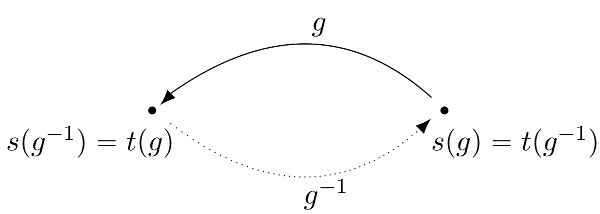
An element $g \in G$ is thought of as an arrow from the object $x = s(g)$ to the object $y = t(g)$:



One therefore usually calls G the *arrow space* of the groupoid, and X its *object space*. As already done so in the axioms, we will usually write gh for the product $m(g, h)$. Whenever we write a product, we are assuming that it is defined. If h is an arrow from $x = s(h)$ to $y = t(h) = s(g)$ and g is an arrow from y to $z = t(g)$, then gh is the composite arrow from x to z:



The inversion map permutes source and target of an arrow:



We will usually denote a groupoid by $\mathsf{G} \underset{s}{\overset{t}{\rightrightarrows}} X$, or if we need to specify all structure maps by

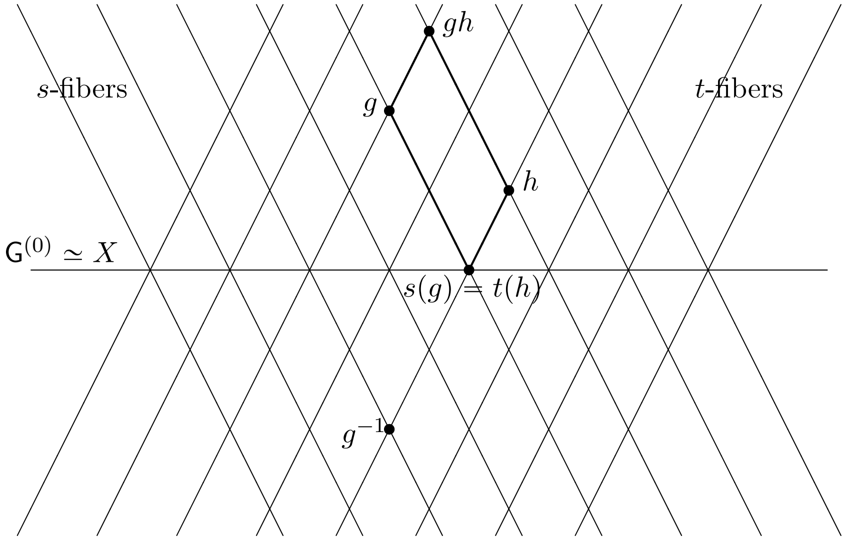
$$\mathsf{G}^{(2)} \xrightarrow{m} \mathsf{G} \underset{s}{\overset{t}{\rightrightarrows}} X \xrightarrow{u} \mathsf{G} \xrightarrow{i} \mathsf{G}\,.$$

By an abuse of notation, we sometimes simply write G for the groupoid above.

A groupoid G gives rise to a hierarchy of sets:

$$\begin{array}{lcl} \mathsf{G}^{(0)} & := & u(X) \simeq X \\ \mathsf{G}^{(1)} & := & \mathsf{G} \\ \mathsf{G}^{(2)} & := & \{(g,h) \in \mathsf{G} \times \mathsf{G} \mid s(g) = t(h)\} \\ \vdots & & \\ \mathsf{G}^{(n)} & := & \{(g_1, \ldots, g_n) \in \mathsf{G}^n \mid s(g_1) = t(g_2), \ldots, s(g_{n-1}) = t(g_n)\} \\ \vdots & & \end{array}$$

The following picture can be useful in visualizing groupoids.



5.1.3 Remark There are various equivalent definitions for groupoids and various ways of thinking of them. As already pointed ourt above, a groupoid G can be viewed as a small category whose objects are the elements of the base set X and whose morphisms are all invertible. Another way to think of a groupoid is as a generalized equivalence relation in which elements of X can be "equivalent in several ways" (see Paragraph 5.1.10). We refer to Brown **?**Brown (2006), as well as **?**, for extensive general discussion of groupoids.

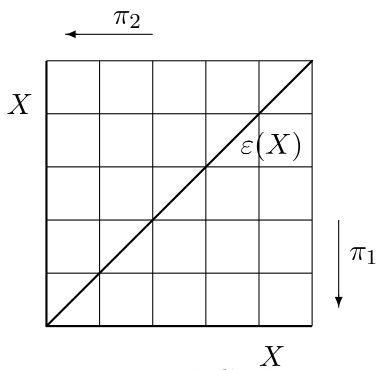
5.1.4 Examples 1. A *group* is a groupoid over a set $X = \{*\}$ with only one element.

2. The *trivial groupoid* over the set X is defined by $\mathsf{G} = X$, and $t = s = \mathrm{id}_X$.

3. Let $\mathsf{G} = X \times X \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} X$ with the groupoid structure defined by

$$\begin{aligned} t(y,x) &:= \mathrm{pr}_2(y,x) = y, \quad s(y,x) := \mathrm{pr}_1(y,x) = x, \\ m\big((z,y),(y,x)\big) &:= (z,y)(y,x) := (z,x), \\ u(x) &:= (x,x), \text{ and } i(y,x) := (y,x)^{-1} := (x,y) \ . \end{aligned}$$

This is often called the *pair groupoid*, or the *coarse groupoid*, or the *Brandt groupoid* after work of Brandt **?**, who is generally credited with introducing the groupoid concept.



5.1.5 Remark Given a groupoid G, choose some $\phi \notin G$. The groupoid multiplication on G extends to a multiplication on the set $G \cup \{\phi\}$ by

$$\begin{aligned} g\phi &= \phi g = \phi \\ gh &= \phi\ , \qquad \text{if } (g,h) \in (G \times G) \backslash \mathsf{G}^{(2)}\ . \end{aligned}$$

The new element ϕ acts as a "receptacle" for any previously undefined product. This endows $G \cup \{\phi\}$ with a *semigroup* structure. A groupoid thus becomes a special kind of semigroup as well.

5.1.6 Definition Given two groupoids G_1 and G_2 over sets X_1 and X_2 respectively, a *morphism of groupoids* is a pair of maps $\mathsf{G}_1 \to \mathsf{G}_2$ and $X_1 \to X_2$ which commute with all the structural functions of G_1 and G_2. We depict a morphism by the following diagram.

$$G_1 G_2{}^{t_1\, t_2}_{s_1\, s_2} X_1 X_2$$

If we consider a groupoid as a special type of category, then a morphism between groupoids is simply a covariant functor between the categories.

There is a natural way to form the *product of groupoids*:

5.1.7 Remark If G_i is a groupoid over X_i for $i = 1, 2$, show that there is a naturally defined direct product groupoid $G_1 \times G_2$ over $X_1 \times X_2$.

disjoint union of groupoids is a groupoid.

Subgroupoids and Orbits

5.1.8 Definition A subset H of a groupoid G over X is called a *subgroupoid* if it is closed under multiplication (when defined) and inversion. Note that

$$h \in \mathsf{H} \Rightarrow h^{-1} \in \mathsf{H} \Rightarrow \text{ both } u(t(h)) \in \mathsf{H} \text{ and } u(s(h)) \in \mathsf{H}\ .$$

Therefore, the subgroupoid H is a groupoid over $t(\mathsf{H}) = s(\mathsf{H})$, which may or may not be all of X. When $t(\mathsf{H}) = s(\mathsf{H}) = X$, H is called a *wide subgroupoid*.

5.1.9 Examples 1. If $\mathsf{G} = X$ is the trivial groupoid, then any subset of G is a subgroupoid, and the only wide subgroupoid is G itself.

2. If $X = \{*\}$ is a one point set, so that G is a group, then the nonempty subgroupoids are the subgroups of G, but the empty set is also a subgroupoid of G.

3. If $\mathsf{G} = X \times X$ is the pair groupoid, then a subgroupoid H is a *relation* on X which is *symmetric* and *transitive*. A wide subgroupoid H is an *equivalence relation*. In general, H is an equivalence relation on the set $t(\mathsf{H}) = s(\mathsf{H}) \subset X$.

5.1.10 For any groupoid G over a set X, there is a morphism

$$G \ ^{(t,s)} X \times X^{t\,\pi_1}_{s\pi_2} X = X$$

from G to the pair groupoid over X. Its image is a wide subgroupoid of $X \times X$, and hence defines an equivalence relation on X. The equivalence classes are called the *orbits* of G in X. In category language, the orbits are the isomorphism classes of the objects of the category. We can also think of a groupoid as an equivalence relation where two elements might be equivalent in different ways, parametrized by the kernel of (t, s). The groupoid further indicates the structure of the set of all ways in which two elements are equivalent.

Inside the groupoid $X \times X$ there is a *diagonal subgroupoid* $\Delta = \{(x, x) \mid x \in X\}$. We call $(t, s)^{-1}(\Delta)$ the *isotropy subgroupoid* of G.

$$(t, s)^{-1}(\Delta) = \{g \in G \mid t(g) = s(g)\} = \bigcup_{x \in X} G_x \ ,$$

where $G_x := \{g \mid t(g) = s(g) = x\}$ is the *isotropy subgroup* of x.

If $x, y \in X$ are in the same orbit, then any element g of

$$G_{x,y} := (t, s)^{-1}(x, y) = \{g \in G \mid t(g) = x \text{ and } s(g) = y\}$$

induces an isomorphism $h \mapsto g^{-1}hg$ from G_x to G_y. On the other hand, the groups G_x and G_y have natural commuting, free transitive actions on $G_{x,y}$, by left and right multiplication, respectively. Consequently, $G_{x,y}$ is isomorphic (as a set) to G_x (and to G_y), but not in a natural way.

A groupoid is called *transitive* if it has just one orbit. The transitive groupoids are the building blocks of groupoids, in the following sense. There is a natural decomposition of the base space of a general groupoid into orbits. Over each orbit there is a transitive groupoid, and the disjoint union of these transitive groupoids is the original groupoid.

5.1.11 Historical Brandt **?** discovered groupoids while studying quadratic forms over the integers. Groupoids also appeared in Galois theory in the description of relations between subfields of a field K via morphisms of K **?**. The isotropy groups of the constructed groupoid turn out to be the Galois groups. Groupoids occur also as generalizations of equivalence relations in the work of Grothendieck on moduli spaces **?** and in the work of Mackey on ergodic theory **?**. For recent applications in these two areas, see Keel and Mori **?** and Connes **?**.

Examples of Groupoids

5.1.12 Example (Fundamental groupoid) Let X be a topological space and let $\mathsf{G} = \pi_1(X)$ be the collection of homotopy classes of continuous paths in X with all possible fixed endpoints. Specifically, if $\gamma : [0,1] \to X$ is a continuous path from $x = \gamma(0)$ to $y = \gamma(1)$, let $[\gamma]$ denote the homotopy class of γ relative to the points x, y. We can define a groupoid

$$\pi_1(X) = \{(x, [\gamma], y) \mid x, y \in X, \gamma \text{ is a continuous path from } x \text{ to } y\} \ ,$$

where multiplication is concatenation of paths. According to our convention, if γ is a path from x to y, the target is $t(x, [\gamma], y) = x$ and the source is $s(x, [\gamma], y) = y$.) The groupoid $\pi_1(X)$ is called the *fundamental groupoid* of X. The orbits of $\pi_1(X)$ are just the path components of X. In fact, for a general groupoid, we will occasionally refer to the orbits as *components*. See Brown's text on algebraic topology Brown (2006) for more on fundamental groupoids.

There are several advantages of the fundamental groupoid over the fundamental group. First notice that the fundamental group sits within the fundamental groupoid as the isotropy subgroup over a single point. The fundamental groupoid does not require a choice of base point and is better suited to study spaces that are not path connected. Additionally, many of the algebraic properties of the fundamental group generalize to the fundamental groupoid, as illustrated in the following result.

5.1.13 Remark Show that the Seifert-Van Kampen theorem on the fundamental group of a union $U \cup V$ can be generalized to groupoids **?**, and that the connectedness condition on $U \cap V$ is then no longer necessary.

5.1.14 Example Let Γ be a group acting on a space X. In the product groupoid $\Gamma \times (X \times X) \simeq X \times \Gamma \times X$ over $\{\text{point}\} \times X \simeq X$, the wide subgroupoid

$$G_\Gamma = \{(x, \gamma, y) \mid x = \gamma \cdot y\}$$

is called the *transformation groupoid* or *action groupoid* of the Γ-action. The orbits and isotropy subgroups of the transformation groupoid are precisely those of the Γ-action.

A groupoid G over X is called *principal* if the morphism $G \xrightarrow{(t,s)} X \times X$ is injective. In this case, G is isomorphic to the image $(t,s)(G)$, which is an equivalence relation on X. The term "principal" comes from the analogy with bundles over topological spaces.

If Γ acts freely on X, then the transformation groupoid G_Γ is principal, and $(t,s)(G_\Gamma)$ is the orbit equivalence relation on X. In passing to the transformation groupoid, we have lost information on the group structure of Γ, as we no longer see *how* Γ acts on the orbits: different free group actions could have the same orbits.

5.1.15 Example Let Γ be a group. There is an interesting ternary operation

$$(x, y, z) \xrightarrow{t} xy^{-1}z \ .$$

It is invariant under left and right translations (check this as an exercise), and it defines 4-tuples $(x, y, z, xy^{-1}z)$ in Γ which play the role of parallelograms. The operation t encodes the affine structure of the group in the sense that, if we know the identity element e, we recover the group operations by setting $x = z = e$ to get the inversion and then $z = e$ to get the multiplication. However, the identity element of Γ cannot be recovered from t.

Denote

$$\begin{aligned} \mathfrak{S}(\Gamma) &= \text{set of subgroups of } \Gamma \\ (\Gamma) &= \text{set of subsets of } \Gamma \text{ closed under } t \,. \end{aligned}$$

5.1.16 Proposition (Γ) *is the set of* cosets *of elements of* $\mathfrak{S}(\Gamma)$.

The sets of right and of left cosets of subgroups of Γ coincide because $gH = (gHg^{-1})g$, for any $g \in G$ and any subgroup $H \leqslant G$.

Prove the proposition above.

We call (Γ) the *Baer groupoid* of Γ, since much of its structure was formulated by Baer **?**. We will next see that the Baer groupoid is a groupoid over $\mathfrak{S}(\Gamma)$.

For $D \in (\Gamma)$, let $t(D) = g^{-1}D$ and $s(D) = Dg^{-1}$ for some $g \in D$. From basic group theory, we know that t and s are maps into $\mathfrak{S}(\Gamma)$ and are independent of the choice of g. Furthermore, we see that $s(D) = gt(D)g^{-1}$ is conjugate to $t(D)$.

$$\begin{array}{c} (\Gamma) \\ \downarrow\downarrow \\ \mathfrak{S}(\Gamma) \end{array}$$

Show that if $s(D_1) = t(D_2)$, *i.e.* $D_1g_1^{-1} = g_2^{-1}D_2$ for any $g_1 \in D_1, g_2 \in D_2$, then the product in this groupoid can be defined by

$$D_1D_2 := g_2D_1 = g_1D_2 = \{gh \mid g \in D_1 \; h \in D_2\} \,.$$

Observe that the orbits of (Γ) are the conjugacy classes of subgroups of Γ. In particular, over a single conjugacy class of subgroups is a transitive groupoid, and thus we see that the Baer groupoid is a refinement of the conjugacy relation on subgroups.

The isotropy subgroup of a subgroup H of Γ consists of all left cosets of H which are also right cosets of H. Any left coset gH is a right coset $(gHg^{-1})g$ of gHg^{-1}. Thus gH is also a right coset of H exactly when $gHg^{-1} = H$, or, equivalently, when $s(gH) = t(gH)$. Thus the isotropy subgroup of H can be identified with $N(H)/H$, where $N(H)$ is the normalizer of H. None

5.1.17 Example Let Γ be a compact connected semisimple Lie group. An interesting conjugacy class of subgroups of Γ is

$$= \{\text{maximal tori of } \Gamma\} \,,$$

where a *maximal torus* of Γ is a subgroup

$$^k \simeq (S^1)^k = S^1 \oplus \cdots \oplus S^1$$

of Γ which is maximal in the sense that there does not exist an $\ell \geqslant k$ such that $^k <^\ell \leqslant \Gamma$ (here, $S^1 \simeq /$ is the circle group). A theorem from Lie group theory (see, for instance, **?**) states that any two maximal tori of a connected Lie group are conjugate, so is an orbit of (Γ). We call the transitive subgroupoid $(\Gamma)|_= \mathcal{W}(\Gamma)$ the *Weyl groupoid* of Γ.

5.1.18 Remarks

- For any maximal torus $\in$, the quotient $N()/$ is the classical Weyl group. The relation between the Weyl groupoid and the Weyl group is analogous to the relation between the fundamental groupoid and the fundamental group.
- There should be relevant applications of Weyl groupoids in the representation theory of a group Γ which is acted on by a second group, or in studying the representations of groups that are not connected.

Groupoids with Structure

groupoid-structure

Ehresmann **?** was the first to endow groupoids with additional structure, as he applied groupoids to his study of foliations. Rather than attempting to describe a general theory of "structured groupoids," we will simply mention some useful special cases.

1. *Topological groupoids:* For a topological groupoid, G and X are required to be topological spaces and all the structure maps must be continuous.

   5.1.19 Examples

   - In the case of a group, this is the same as the concept of topological group.
   - The pair groupoid of a topological space has a natural topological structure derived from the product topology on $X \times X$.

   For analyzing topological groupoids, it is useful to impose certain further axioms on G and X. For a more complete discussion, see **?**. Here is a sampling of commonly used axioms:

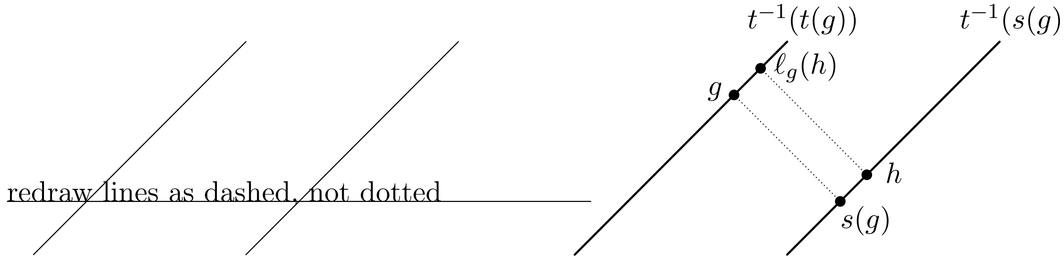
   a) $\mathsf{G}^{(0)} \simeq X$ is locally compact and Hausdorff.

   b) The t- and s-fibers are locally compact and Hausdorff.

   c) There is a countable family of compact Hausdorff subsets of G whose interiors form a basis for the topology.

   d) G admits a *Haar system*, that is, admits a family of measures on the t-fibers which is invariant under left translations. For any $g \in G$, left translation by g is a map between t fibers

$$\begin{array}{rcl} t^{-1}(s(g)) & \longrightarrow & t^{-1}(t(g)) \\ h & \overset{\ell_g}{\longmapsto} & gh\ . \end{array}$$



5.1.20 Example For the pair groupoid, each fiber can be identified with the base space X. A family of measures is invariant under translation if and only if the measure is the same on each fiber. Hence, a Haar system on a pair groupoid corresponds to a measure on X.

2. *Measurable groupoids:* These groupoids, also called *Borel groupoids*, come equipped with a σ-algebra of sets and a distinguished subalgebra (called the null sets); see **??**. On each t-fiber, there is a *measure class*, which is simply a measure defined up to multiplication by an invertible measurable function.

3. *Lie groupoids* or *differentiable groupoids:* The groupoid G and the base space X are manifolds and all the structure maps are smooth. It is *not* assumed that G is Hausdorff, but only that $\mathsf{G}^{(0)} \simeq X$ is a Hausdorff manifold and closed in G.[1] Thus we can require that the identity section be smooth. Recall that multiplication is defined as a map on $\mathsf{G}^{(2)} \subseteq G$. To require that multiplication be smooth, first $\mathsf{G}^{(2)}$ needs to be a smooth manifold. It is convenient to make the stronger assumption that the map t (or s) be a submersion.

   Show that the following conditions are equivalent:

   (a) t is a submersion,

   (b) s is a submersion,

   (c) the map (t, s) to the pair groupoid is transverse to the diagonal.

   Note that t is the identity map along $\mathsf{G}^{(0)}$ and is thus automatically a submersion near $\mathsf{G}^{(0)}$.

   Show that t is a submersion on an open and closed subset of G.

   Thus, we could drop the submersion assumption on t, by assuming that G is connected.

4. *Bundles of groups:* A groupoid for which $t = s$ is called a bundle of groups. This is not necessarily a trivial bundle, or even a locally trivial bundle in the topological case, as the fibers need not be isomorphic as groups or as topological spaces. The orbits are the individual points of the base space, and the isotropy subgroupoids are the fiber groups of the bundle.

[1]Throughout these notes, a manifold is assumed to be Hausdorff, *unless* it is a groupoid.

The Holonomy Groupoid of a Foliation

Let X be a (Hausdorff) manifold. Let $F \subseteq TX$ be an integrable subbundle, and the corresponding foliation (is the decomposition of X into maximal integral manifolds called *leaves*). The notion of holonomy can be described as follows. An F-path is a path in X whose tangent vectors lie within F. Suppose that $\gamma : [0,1] \to$ is an F-path along a leaf . Let $N_{\gamma(0)}$ and $N_{\gamma(1)}$ be cross-sections for the spaces of leaves near $\gamma(0)$ and $\gamma(1)$, respectively, *i.e.* they are two small transversal manifolds to the foliation at the end points of γ. There is an F-path near γ from each point near $\gamma(0)$ in $N_{\gamma(0)}$ to a uniquely determined point in $N_{\gamma(1)}$. This defines a local diffeomorphism between the two leaf spaces. The *holonomy* of γ is defined to be the *germ*, or direct limit, of such diffeomorphisms, between the local leaf spaces $N_{\gamma(0)}$ and $N_{\gamma(1)}$.

The notion of holonomy allows us to define an equivalence relation on the set of F-paths from x to y in X. Let $[\gamma]_H$ denote the equivalence class of γ under the relation that two paths are equivalent if they have the same holonomy.

The *holonomy groupoid* **?**, also called the *graph* of the foliation **?**, is

$$H() = \{(x, [\gamma]_H, y) \mid x, y \in X, \gamma \text{ is an } F\text{-path from } x \text{ to } y\} .$$

Given a foliation , there are two other related groupoids obtained by changing the equivalence relation on paths:

1. The *-pair groupoid* – This groupoid is the equivalence relation for which the equivalence classes are the leaves of , *i.e.* we consider any two F-paths between $x, y \in$ to be equivalent.
2. The *-fundamental groupoid* – For this groupoid, two F-paths between x, y are equivalent if and only if they are *F-homotopic*, that is, homotopic within the set of all F-paths. Let $[\gamma]_F$ denote the equivalence class of γ under F-homotopy. The set of this groupoid is

$$\Pi() = \{(x, [\gamma]_F, y) \mid x, y \in X, \gamma \text{ is an } F\text{-path from } x \text{ to } y\} .$$

If two paths γ_1, γ_2 are F-homotopic with fixed endpoints, then they give the same holonomy, so the holonomy groupoid is intermediate between the -pair groupoid and the -fundamental groupoid:

$$[\gamma_1]_F = [\gamma_2]_F \implies [\gamma_1]_H = [\gamma_2]_H .$$

The pair groupoid may not be a manifold. With suitably defined differentiable structures, though, we have:

$H()$ and $\Pi()$ are (not necessarily Hausdorff) Lie groupoids.

For a nice proof of this theorem, and a comparison of the two groupoids, see **?**. Further information can be found in **?**.

Compare the -pair groupoid, the holonomy groupoid of , and the -fundamental groupoid for the Möbius band and the Reeb foliation, as described below.

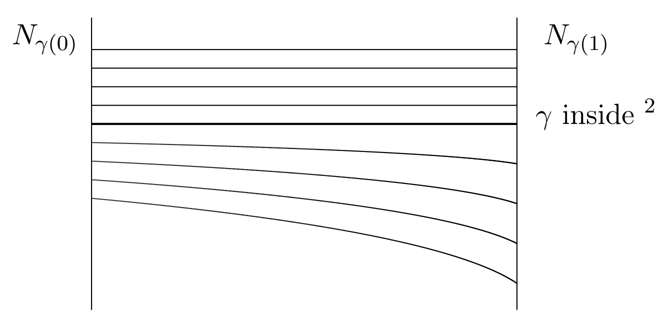
1. The *Möbius band*. Take the quotient of the unit square $[0,1] \times [0,1]$ by the relation $(1,x) \sim (0,1-x)$. Define the leaves of to be images of the horizontal strips $\{(x,y) \mid y = \text{constant}\}$.

2. The *Reeb foliation* **?**. Consider the family of curves $x = c + \sec y$ on the strip $-\pi/2 < y < \pi/2$ in the xy-plane. If we revolve about the axis $y = 0$, then this defines a foliation of the solid cylinder by planes. Noting that the foliation is invariant under translation, we see that this defines a foliation of the open solid torus $D^2 \times S^1$ by planes. The foliation is smooth because its restriction to the xy-plane is defined by the 1-form $\cos^2 y\, dx + \sin y\, dy$, which is smooth even when $y = \pm\frac{\pi}{2}$. We close the solid torus by adding one exceptional leaf – the ^{2} boundary.

   Let t be a vanishing cycle on ^{2}, that is, $[t] \in \pi_1(^2)$ generates the kernel of the natural map $\pi_1(^2) \to \pi_1(D^2 \times S^1)$. Although t is not null-homotopic on the exceptional leaf, any perturbation of t to a nearby leaf results in a curve that is F-homotopically trivial. On the other hand, the transverse curve (the cycle given by $(c,y) \in D^2 \times S^1$ for some fixed $c \in \partial D^2$) cannot be pushed onto any of the nearby leaves.

   A basic exercise in topology shows us that we can glue two solid tori together so that the resulting manifold is the 3-sphere S^3. For this gluing, the transverse cycle of one torus is the vanishing cycle of the other. (If we instead glued the two vanishing cycles and the two transverse cycles together, we would obtain $S^2 \times S^1$.)

   It is interesting to compute the holonomy on each side of the gluing ^{2}. Each of the two basic cycles in ^{2} has trivial holonomy on one of its sides (holonomy given by the germ of the identity diffeomorphism), and *non*-trivial holonomy on the other side (given by the germ of an expanding diffeomorphism).



   This provides an example of *one-sided holonomy*, a phenomenon that cannot happen for real analytic maps. The leaf space of this foliation is not Hausdorff; in fact, any function constant on the leaves must be constant on all of S^3, since all leaves come arbitrarily close to the exceptional leaf ^{2}. This foliation and its holonomy provided the inspiration for the following theorems.

   [Haefliger **?**] S^3 has no real analytic foliation of codimension-1.

   [Novikov **?**] Every codimension-1 foliation of S^3 has a compact leaf that is a torus.

A. Miscellanea from Linear Algebra

A.0.1 Definition Let $\Bbbk$ be a field and V a vector space over $\Bbbk$. Recall that a map $B : V \times V \to \Bbbk$ is called a *bilinear form*, if the following two axioms hold true:

(BF1) B is linear in its first argument, i.e.

$$B(v + v', w) = B(v, w) + B(v', w) \text{ \& } B(av, w) = aB(v, w) \text{ for all } v, v', w \in V,\ a \in \Bbbk\,.$$

(BF2) B is linear in its second argument, i.e.

$$B(v, w + w') = B(v, w) + B(v, w') \text{ \& } B(v, aw) = aB(v, w) \text{ for all } v, w, w' \in V,\ a \in \Bbbk\,.$$

A bilinear form $B : V \times V \to \Bbbk$ is called *symmetric*, if

(BF3) $B(v, w) = B(w, v)$ for all $v, w \in V$,

and *anti-symmetric* or *skew-symmetric*, if

(BF4) $B(v, w) = -B(w, v)$ for all $v, w \in V$.

A.0.2 Definition Let V be a complex vector space. Recall that a map $h : V \times V \to \mathbb{C}$ is called a *sesquilinear form*, if the following two conditions are satisfied:

(SF1) h is linear in its first argument, i.e.

$$h(v + v', w) = h(v, w) + h(v', w) \text{ \& } h(zv, w) = zh(v, w) \text{ for all } v, w, w' \in V,\ z \in \mathbb{C}\,.$$

(SF2) h is antilinear in its second argument, i.e.

$$h(v, w + w') = h(v, w) + h(v, w') \text{ \& } h(v, zw) = \overline{z}h(v, w) \text{ for all } v, w, w' \in V,\ z \in \mathbb{C}\,.$$

A sesquilinear form $h : V \times V \to \Bbbk$ is called a *hermitian form*, if in addition

(SF3) $h(v, w) = \overline{h(w, v)}$ for all $v, w \in V$.

A.1. Multilinear algebra

The tensor product

A.1.1 Let V and W be two vector spaces over the field $\Bbbk$. Denote by $F(V,W)$ the free vector space over the cartesian product $V \times W$ that means let

$$F(V,W) := \Bbbk^{(V\times W)} := \left\{\lambda : V \times W \to \Bbbk \mid \lambda^{-1}(0) \text{ is finite}\right\} .$$

A basis of $F(V,W)$ is given by the family $\left\{[v,w]\right\}_{(v,w)\in V\times W}$ of elements

$$[v,w] : V \times W \to \Bbbk,\ (v',w') \mapsto \begin{cases} 1 & \text{if } v = v' \text{ and } w = w, \\ 0 & \text{else .} \end{cases}$$

Now define $R(V,W)$ as the subspace of $F(V,W)$ generated by the set of elements

$$[v+v',w]-[v,w]-[v',w],\ [rv,w]-r[v,w],\ [v,w+w']-[v,w]-[v,w'],\ [v,rw]-r[v,w],$$

where $v,v' \in V$, $w,w' \in W$, and $r \in \Bbbk$. The quotient space $T(V,W) := F(V,W)/R(V,W)$ then is a vector space over $\Bbbk$, and the canonical map

$$i : V \times W \to T(V,W),\ (v,w) \mapsto v \otimes w := [v,w] + R(V,W)$$

bilinear. We will show that $T(V,W)$ together with i is a tensor product in the sense of the subsequent definition.

A.1.2 Definition Given two vector spaces V,W over a field $\Bbbk$ a vector space $V \otimes W$ together with $\Bbbk$-bilinear map $i : V \times W \to V \otimes W$ is called a *tensor product* of the vector spaces V and W, if it satisfies the following universal property:

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